

ISOMETRIES FOR THE CARATHÉODORY METRIC

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1. INTRODUCTION

The following problem has been studied by many authors. Let D_1 and D_2 be two bounded domains in complex Banach spaces and let $f: D_1 \rightarrow D_2$ be a holomorphic map such that $f'(a)$ is a surjective isometry for the Carathéodory infinitesimal metric at a point a of D_1 . The problem is to know whether f is an analytic isomorphism of D_1 onto D_2 . For example, J.-P. Vigué [8] proved this is the case when D_1 and D_2 are two bounded domains in \mathbb{C}^n and D_1 is convex. Similar results have been obtained when D_2 is convex using the Kobayashi infinitesimal metric (I. Graham [3] and L. Belkhchicha [1]). We have to remark that all these results are based on the theorem of L. Lempert ([5] et [6]; one can also consult M. Jarnicki and P. Pflug [4]) on the equality of Kobayashi and Carathéodory metrics on a bounded convex domain in \mathbb{C}^n . J.-P. Vigué [9] proved the first results on this subject in the case of bounded domains in complex Banach spaces.

Now, we can study the same problem dropping the hypothesis that $f'(a)$ is surjective. So, we only suppose that $f'(a)$ is an isometry for the Carathéodory infinitesimal metric. Does this imply that $f(D_1)$ is a complex analytic closed submanifold of D_2 and that f is an analytic isomorphism of D_1 onto $f(D_1)$?

Some results have been obtained by J.-P. Vigué [10] and P. Mazet [7] assuming that D_1 and D_2 are open unit balls in complex Banach spaces, that $a = 0$, and that the image of $f'(0)$ contains enough complex extremal points of the boundary of D_2 . Under these hypotheses they proved that f is linear equal to $f'(0)$. This result shows that $f(D_1)$ is an analytic submanifold of D_2 and that f is an analytic isomorphism of D_1 onto $f(D_1)$.

Of course, if we do not suppose the existence of complex extremal points in the image of $f'(0)$, the map f has no reason to be linear. However, one can hope that $f(D_1)$ still is a complex analytic submanifold of D_2 . In this paper we shall be able to prove such a result for maps of unit balls of complex Banach spaces, under some additional hypotheses on the Banach spaces involved.

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2. THE MAIN RESULTS

We shall prove the following theorem:

Theorem 1. *Let $(E_j, \|\cdot\|_j)$ be complex Banach spaces and let $B_j = \{x \in E_j \mid \|x\|_j < 1\}$, for $j = 1, 2$. Let $f: B_1 \rightarrow B_2$ be a holomorphic mapping with $f(0) = 0$ and $\|f'(0)(X)\|_2 = \|X\|_1$ for all $X \in E_1$. Then the following statements are equivalent :*

- (1) *there exists a direct decomposition $E_2 = f'(0)(E_1) \oplus F$ such that the corresponding projection $\pi: E_2 \rightarrow f'(0)(E_1)$ has norm 1;*
- (2) *$f(B_1)$ is a closed complex direct submanifold of B_2 , the map f is a biholomorphism of B_1 onto $f(B_1)$, and there exists a holomorphic retraction of B_2 onto $f(B_1)$.*

To apply this theorem, we give the following definition:

Definition 1. *We say that a pair (E_1, E_2) of complex Banach spaces has the property (V) if for every linear isometry $L: E_1 \rightarrow E_2$ there exists a direct decomposition $E_2 = L(E_1) \oplus F$ such that the corresponding projection $\pi: E_2 \rightarrow L(E_1)$ has norm 1.*

From theorem 1 and definition 1, we deduce the following

Theorem 2. *Assume that the pair (E_1, E_2) of complex Banach spaces has the property (V), and let B_1 and B_2 be their open unit balls. Let $f: B_1 \rightarrow B_2$ be a holomorphic map such that*

- (1) *$f(0) = 0$, and $f'(0)$ is an isometry for the Carathéodory infinitesimal metric,*
- or*
- (2) *B_1 and B_2 are homogeneous, and there exists $a \in B_1$ such that $f'(a)$ is an isometry for the Carathéodory infinitesimal metric.*

Then $f(B_1)$ is a closed complex direct submanifold of B_2 , the map f is a biholomorphism of B_1 onto $f(B_1)$, and there exists a holomorphic retraction of B_2 onto $f(B_1)$.

Now, we clearly need examples of pairs of complex Banach spaces satisfying property (V). The first (easy) example is given by Hilbert spaces.

Proposition 1. *Let E_2 a complex Hilbert space. Then, for every complex Banach space E_1 the pair (E_1, E_2) has property (V).*

More interesting is the following theorem:

Theorem 3. *Let I be a set and let $l^\infty(I)$ be the complex Banach space of bounded sequences indexed by I , with the usual norm. Let E_2 be any Banach space. Then, the pair $(l^\infty(I), E_2)$ has property (V).*

Other pairs enjoying property (V) can be constructed using suitable subspaces of $l^\infty(I)$. For instance, let $c_0(I) \subset l^\infty(I)$ be the subspace given by the elements $(a_i)_{i \in I} \in l^\infty(I)$ such that for every $\varepsilon > 0$ there exists a finite subset $K \subseteq I$ so that $|a_i| < \varepsilon$ when $i \notin K$. Then:

Theorem 4. *For any sets I, J the pair $(c_0(I), c_0(J))$ has property (V).*

Applying Theorem 2 and 3 with I finite, we get in particular a new result in the finite-dimensional case:

Corollary 1. *Let $f: \Delta^n \rightarrow D$ be a holomorphic map between a polydisk $\Delta^n \subset \mathbb{C}^n$ and an open convex circular bounded domain $D \subset \mathbb{C}^N$ (i.e., D is the unit ball for a suitable norm in \mathbb{C}^N). We also assume $n \leq N$, and that D is homogeneous (for instance, $D = \Delta^N, B^N$ or a bounded symmetric domain). Assume that there exists $a \in \Delta^n$ such that $f'(a)$ is an isometry for the Carathéodory infinitesimal metrics. Then $f(\Delta^n)$ is a closed complex submanifold of D , the map f is a biholomorphism onto its image, and $f(\Delta^n)$ is a holomorphic retract of D .*

Before proving these results, we need to recall some facts.

3. SOME CLASSICAL RESULTS

The definition and the main properties of Carathéodory and Kobayashi infinitesimal metrics E_D and F_D on a bounded domain D are given in the book of T. Franzoni et E. Vesentini [2] (see also the book of M. Jarnicki and P. Pflug [4]).

Let B be the open unit ball of a complex Banach space E . It is well known that

$$E_B(0, x) = F_B(0, x) = \|x\|.$$

Furthermore, every biholomorphism $f: D_1 \rightarrow D_2$ between domains in complex Banach spaces is an isometry for the Carathéodory and Kobayashi infinitesimal metrics.

Finally, let us recall that, the open unit balls B of the complex Banach spaces $c_0(I)$ and $l^\infty(I)$ are homogeneous. Indeed, it is easy to check that, for every $a \in B$, the map $\varphi_a: B \rightarrow B$ given by

$$\forall i \in I \quad \varphi_a(f)_i = \frac{f_i + a_i}{1 + \overline{a_i} f_i},$$

is an analytic automorphism of B .

Another example of homogeneous unit ball is given by the open unit ball B of the space $C(S, \mathbb{C})$ of continuous complex functions on a

compact space S , because for every $a \in B$ the map $\varphi_a: B \rightarrow B$ given by

$$\varphi_a(f) = \frac{f + a}{1 + \bar{a}f}$$

is a biholomorphism of B .

4. PROOF OF THEOREMS 1 AND 2

To begin, let us prove theorem 1.

Proof of Theorem 1. First, if $r: B_2 \rightarrow f(B_1)$ is a holomorphic retraction, $r'(0)$ is a projection of norm ≤ 1 for the Carathéodory infinitesimal metrics, and,

as the Carathéodory infinitesimal metric at the origin is equal to the given norm, we get $\|r'(0)\| = 1$. This proves that (2) implies (1).

To prove that (1) implies (2), let us consider

$$\varphi = \pi \circ f: B_1 \rightarrow f'(0)(E_1) .$$

We have $\varphi(0) = 0$, $\varphi(B_1) \subseteq f'(0)(E_1) \cap B_2$ (because π has norm 1), and $\varphi'(0) = \pi \circ f'(0) = f'(0)$. So $\varphi'(0)$ is a linear isometry from E_1 onto $f'(0)(E_1)$. Using Cartan's uniqueness theorem (see [2]), one easily proves that φ is a linear isometry from B_1 onto $B'_2 = f'(0)(E_1) \cap B_2$.

Finally, let $\psi: B'_2 \rightarrow F$ be defined by

$$\psi(y) = (\text{id} - \pi)(f(\varphi^{-1}(y))) .$$

Then the set $f(B_1)$ is the graph of ψ , the map $(\pi, \psi \circ \pi): B_2 \rightarrow f(B_1)$ is a holomorphic retraction of B_2 onto $f(B_1)$, and $\varphi^{-1} \circ \pi|_{f(B_1)}: f(B_1) \rightarrow B_1$ is a holomorphic inverse of f , and the theorem is proved. \square

Now, we can prove Theorem 2.

Proof of Theorem 2. First, let us remark that, in case (2), by pre-composing f with an analytic automorphism of B_1 and post-composing it with an analytic automorphism of B_2 , we can assume that $f(0) = 0$ and that 0 is precisely the point a such that $f'(0)$ is an isometry for the Carathéodory infinitesimal metrics. Thus without loss of generality in both cases we can assume that $f'(0)$ is an isometry for the norms of E_1 and E_2 .

Since (E_1, E_2) satisfies the property (V), there exists a direct decomposition $E_2 = f'(0)(E_1) \oplus F$ such that the corresponding projection $\pi: E_2 \rightarrow f'(0)(E_1)$ has norm 1 and we can apply Theorem 1. \square

5. EXAMPLES OF PAIR OF BANACH SPACES WITH PROPERTY (V)

Now we have to give examples of pair of complex Banach spaces satisfying property (V). Proposition 1 (the case of Hilbert spaces) is easy and left as an exercise. Let us now give the

Proof of Theorem 3. We suppose that $E_1 = l^\infty(I)$ and we consider an isometry $L: \ell^\infty(I) \rightarrow E_2$. Let $G: L(E_1) \rightarrow l^\infty(I)$ be the inverse of L . So, G is a linear map of norm 1; for every $i \in I$, let G_i be the i -component of G . Then G_i is a linear form from $L(E_1)$ to \mathbb{C} of norm 1. By the Hahn-Banach Theorem, we can extend G_i to a linear form $H_i: E_2 \rightarrow \mathbb{C}$ still of norm 1. Setting $H = (H_i)_{i \in I}$ we obtain a linear map $H: E_2 \rightarrow l^\infty(I)$ of norm 1 extending G . Then it is clear that $L \circ H$ is a projection of E_2 onto $L(l^\infty(I))$ of norm 1, and taking $F = \text{Ker}(L \circ H)$ the theorem is proved. \square

Proof of Theorem 4. Let $L: c_0(I) \rightarrow c_0(J)$ be an isometry, and let $(e^k)_{k \in I}$ be the canonical basis of $c_0(I)$. Since L is an isometry, for every $k \in I$ there exists $j(k) \in J$ such that $|L(e^k)_{j(k)}| = 1$. Now, if we consider an element $v = (v_i)_{i \in I}$ of $c_0(I)$ such that $v_k = 0$, then $L(v)_{j(k)} = 0$. In fact, suppose that $L(v)_{j(k)} \neq 0$. For $\lambda \in \mathbb{C}$ small enough, we have $\|e^k + \lambda v\| = 1$. But

$$L(e^k + \lambda v)_{j(k)} = L(e^k)_{j(k)} + \lambda L(v)_{j(k)} = e^{i\theta} + \lambda L(v)_{j(k)} .$$

Therefore if $L(v)_{j(k)} \neq 0$, there exists $\lambda \in \mathbb{C}$ small enough such that the modulus of $L(e^k + \lambda v)_{j(k)}$ is greater than 1, and thus $\|L(e^k + \lambda v)\| > 1$, contradiction. It follows that the map $k \mapsto j(k)$ is injective.

Let $M = \{j(k) \mid k \in I\} \subseteq J$, and let $\pi: c_0(J) \rightarrow c_0(M)$ be the canonical projection. The previous argument shows

that $\pi \circ L(e^k) = \lambda_k e^{j(k)}$ with $|\lambda_k| = 1$ for all $k \in I$; it is then easy

to check that $\varphi = \pi \circ L: c_0(I) \rightarrow c_0(M)$ is a linear surjective isometry, and that $L \circ \varphi^{-1} \circ \pi: c_0(J) \rightarrow L(c_0(I))$ is a linear projection of norm 1 of $c_0(J)$ onto $L(c_0(I))$, as required. \square

It might be interesting to remark that the same proof yields that a pair (E_1, E_2) of complex Banach spaces satisfies property (V) if each E_j has a Schauder basis (e_j^k) such that

$$\left\| \sum_k \lambda_k e_j^k \right\|_{E_j} = \sup_k |\lambda_k| .$$

6. FINAL REMARKS

Not all pairs of complex Banach spaces have property (V); so we do not know whether Theorem 2 holds in general.

For example, the Banach spaces $c_0(\mathbb{N})$ is not complemented in $l^\infty(\mathbb{N})$, and so the pair $(c_0(\mathbb{N}), l^\infty(\mathbb{N}))$ does not have property (V).

It is also possible to build finite dimensional examples. Take $E_2 = (\mathbb{C}^3, \|\cdot\|_\infty)$, so that the unit ball of E_2 is the open polydisk Δ^3 . If $L: \mathbb{C}^2 \rightarrow \mathbb{C}^3$ is given by $L(x, y) = (x, y, x + y)$, then $B = L^{-1}(\Delta^3)$ is the open unit ball in \mathbb{C}^2 for a norm $\|\cdot\|$; set $E_1 = (\mathbb{C}^2, \|\cdot\|)$. We claim that the pair (E_1, E_2) does not satisfy property (V). By construction, $L: E_1 \rightarrow E_2$ is a linear isometry. The set $(1, 0, 1) + (\{0\} \times \Delta \times \{0\})$ is contained in the boundary of Δ^3 . Since $(1, 0, 1) \in L(\partial B)$, it is easy to check that if there exists a projection π of norm 1 from \mathbb{C}^3 onto $L(\mathbb{C}^2)$, then π must vanish on $\{0\} \times \mathbb{C} \times \{0\}$. Considering the point $(0, 1, 1)$, we analogously see that π must vanish on $\Delta \times \{0\} \times \{0\}$; and thus such a projection cannot exist.

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